

CONVOLUTION ESTIMATES AND MODEL SURFACES OF LOW CODIMENSION

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ABSTRACT. For $k \geq d/2$ we give examples of measures on k -surfaces in \mathbb{R}^d . These measures satisfy convolution estimates which are nearly optimal.

Suppose that S is a smooth k -dimensional surface in \mathbb{R}^d and that μ is a smooth positive Borel measure on S . Suppose further that μ satisfies the convolution estimate

$$(1) \quad \|\mu * f\|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)},$$

where the norms are computed using Lebesgue measure m_d on \mathbb{R}^d . Then it is well-known that $p \leq q$. Convolution with the characteristic function of a small ball shows that $(1/p, 1/q)$ must lie in the triangle $\Delta(k, d)$ with vertices $(0, 0)$, $(1, 1)$, and $(d/(2d - k), (d - k)/(2d - k))$. And a result of Ricci ([5]), which extends an observation of Carbery and Christ, shows that if $k(k + 3) < 2d$, then (1) also implies that

$$(2) \quad \frac{1}{p} - \frac{1}{q} \leq \frac{2k}{6d - k^2 - 5k}.$$

Let $\mathcal{T}(k, d)$ be $\Delta(k, d)$ if $k(k + 3) \geq 2d$ and the subset of $\Delta(k, d)$ defined by (2) if $k(k + 3) < 2d$. Suppose now that S has the form

$$(3) \quad \{(y; \Phi_1(y), \Phi_2(y), \dots, \Phi_l(y)) : y \in G\}$$

where G is a nonempty open subset of \mathbb{R}^k , where $l = d - k$, and where the functions $\Phi_j : \mathbb{R}^k \rightarrow \mathbb{R}$ are homogeneous polynomials. Let μ be the measure on S induced by m_k on G . Then we will say that S is a *model surface* if (1) holds whenever $(1/p, 1/q)$ lies in the interior of $\mathcal{T}(k, d)$.

Examples:

- (i) the paraboloids $\{(y; |y|^2) : y \in \mathbb{R}^{d-1}, |y| < 1\}$ (see, e.g., pp. 370–371 in [6]);
- (ii) the moment curves $\{(y; y^2, \dots, y^d) : 0 < y < 1\}$ (see [1]);
- (iii) the monomial surfaces $\{(y; \Phi_1(y), \dots, \Phi_l(y)) : y \in \mathbb{R}^k, |y| < 1\}$ where $l = k + \frac{k(k-1)}{2}$ and the functions Φ_j are the distinct quadratic monomials (see [5]);

Date: June, 2007.

1991 *Mathematics Subject Classification.* 42B10.

Key words and phrases. measure, convolution estimate.

The author was supported in part by NSF grant DMS-0552041.

(iv) the 3-surface $\{(y_1, y_2, y_3; y_1^2 + y_2^2, y_2^2 + y_3^2) : 0 < y_j < 1\}$ in \mathbb{R}^5 (see [4]);

(v) certain surfaces of the form $\{(y; \Phi_1(y), \dots, \Phi_l(y)) : y \in \mathbb{R}^k, |y| < 1\}$ where $l = k$ (see [2]).

Of course most polynomial surfaces S of the form (3) are not model surfaces in our sense: the convolution requirement rules out degeneracies which result from the presence of “flatness” or the lack of “curvature”. When $k = 1$ or $k = d - 1$ there are obvious and simple technical interpretations of “curvature”. In a few other cases there are technical interpretations which are neither obvious nor simple. For example, when $k = 2$ and $d = 4$ the interpretation is that

$$(\Phi_{1y_1y_1} \Phi_{2y_1y_2} - \Phi_{2y_1y_1} \Phi_{1y_1y_2})(\Phi_{2y_2y_2} \Phi_{1y_1y_2} - \Phi_{2y_1y_2} \Phi_{1y_2y_2}) - ((\Phi_{1y_1y_1} \Phi_{2y_2y_2} - \Phi_{2y_1y_1} \Phi_{1y_2y_2}))^2$$

not vanish. At any rate, the examples mentioned above, along with certain of their Cartesian products, constitute a fairly complete list of the known model surfaces. The aim of this note is to extend that list by providing examples of model surfaces whenever $k \geq \frac{d}{2}$.

Fix positive integers k and l with $1 \leq l \leq k$ and put $d = k + l$. Let $\mathcal{C} = [c_i^j]$ be a k by l matrix of real numbers. For $1 \leq j \leq l$ define bilinear forms $L_j : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}$ by

$$L_j(x, y) = \sum_{i=1}^k c_i^j x_i y_i$$

and put $\Phi_j(y) = L_j(y, y)$. We will say that \mathcal{C} satisfies condition (*) if every l by l submatrix of \mathcal{C} is nonsingular.

Theorem. *With the Φ_j as above, with $G = B(0, 1)$, and with S given by (3), suppose that (*) holds. Then S is a model surface.*

Proof of theorem: Since $k(k+3) \geq 2d$, it is required to establish (1) whenever $(\frac{1}{p}, \frac{1}{q})$ lies in the interior of $\Delta(k, d)$. With $q_0 = \frac{2d-k}{d-k}$, an interpolation argument shows that it suffices to prove that

$$(4) \quad \|\mu * \chi_E\|_{q_0} \leq C(p) m_d(E)^{1/p}$$

for measurable $E \subset \mathbb{R}^d$ and $p > \frac{2d-k}{d}$. And, since μ has compact support, we can also assume that $E \subset B(0, 1)$. For such E , (4) will follow, as in [4], from the auxiliary inequality (6) below. Thus, writing $\Phi(y) = (y; \Phi_1(y), \dots, \Phi_l(y))$,

$$\begin{aligned} (5) \quad & \|\mu * \chi_E\|_{q_0}^{q_0} \\ &= \int_{\mathbb{R}^d} \int_{B(0,1)} \chi_E(z - \Phi(x)) dm_k(x) \left(\int_{B(0,1)} \chi_E(z - \Phi(y)) dm_k(y) \right)^{q_0-1} dm_d(z) \\ &= \int_{\mathbb{R}^d} \chi_E(z) \int_{B(0,1)} \left(\int_{B(0,1)} \chi_E(z + \Phi(x) - \Phi(y)) dm_k(y) \right)^{q_0-1} dm_k(x) dm_d(z). \end{aligned}$$

Now assume, for the moment, the inequality

$$(6) \quad \left(\int_{B(0,1)} \left[\int_{B(0,1)} \chi_{\tilde{E}}(\Phi(x) - \Phi(y)) dm_k(y) \right]^{d/l} dm_k(x) \right)^{l/d} \leq C(\tilde{p}) m_d(\tilde{E})^{1/\tilde{p}}$$

for $\tilde{p} > \frac{d}{k}$ and $\tilde{E} \subset B(0, 2)$. Since $q_0 - 1 = \frac{d}{l}$, (5) and (6) yield

$$\|\mu * \chi_E\|_{q_0} \leq C(\tilde{p}) m_d(E)^{(1+\frac{1}{\tilde{p}}\frac{d}{l})\frac{1}{q_0}}.$$

If $\frac{1}{p} = (1 + \frac{1}{\tilde{p}}\frac{d}{l})\frac{1}{q_0}$, then $\tilde{p} > \frac{d}{k}$ if and only if $p > \frac{2d-k}{d}$. Thus, as claimed, (4) will follow from (6). Now (6) is equivalent to the inequality, for nonnegative

$$(7) \quad \int_{B(0,1)} \int_{B(0,1)} f(x) \chi_{\tilde{E}}(x - y; \sum c_i^1(x_i^2 - y_i^2), \dots, \sum c_i^l(x_i^2 - y_i^2)) dm_k(y) dm_k(x) \leq C(\tilde{p}) \|f\|_{L^{d/k}(\mathbb{R}^k)} m_d(\tilde{E})^{1/\tilde{p}},$$

where \sum means $\sum_{i=1}^k$. In the y -integral we change variables to obtain

$$\int_{B(0,1)} \int_{B(0,1)} f(x) \chi_{\tilde{E}}(y; \sum c_i^1(2x_i y_i - y_i^2), \dots, \sum c_i^l(2x_i y_i - y_i^2)) dm_k(y) dm_k(x).$$

If $E \subset \mathbb{R}^d$ is defined by

$$\chi_E(y_1, \dots, y_k; u_1, \dots, u_l) = \chi_{\tilde{E}}(y_1, \dots, y_k, 2u_1 - \sum c_i^1 y_i^2, \dots, 2u_l - \sum c_i^l y_i^2),$$

then $m_d(E) = 2^{-l} m_d(\tilde{E})$ and the left hand side of (7) may be written

$$\int_{B(0,1)} \int_{B(0,1)} f(x) \chi_E(y; L_1(x, y), \dots, L_l(x, y)) dm_k(y) dm_k(x).$$

Thus (7) will follow from

$$(8) \quad \int_{B(0,1)} \int_{B(0,1)} f(x) \chi_E(y; L_1(x, y), \dots, L_l(x, y)) dm_k(y) dm_k(x) \leq C \|f\|_{L^{d/k}(\mathbb{R}^k)} m_d(E)^{1/\tilde{p}}$$

whenever f is nonnegative, $\tilde{p} > \frac{d}{k}$, and $E \subset B(0, 2)$. (The constant C will depend on \tilde{p} and \mathcal{C} .) For an multi-index $\mathbf{n} = (n_1, \dots, n_k)$ we will write $\{|y_i| \sim 2^{n_i}\}$ to stand for the set of $y \in \mathbb{R}^k$ for which the k inequalities $2^{n_i} \leq |y_i| < 2^{n_i+1}$ hold. Our main task will be to establish the estimate

$$(9) \quad \int_{\mathbb{R}^k} \int_{\{|y_i| \sim 1\}} f(x) \chi_E(y; L_1(x, y), \dots, L_l(x, y)) dm_k(y) dm_k(x) \leq C \|f\|_{L^{d/k}(\mathbb{R}^k)} m_d(E)^{k/d}$$

for all nonnegative f and $E \subset \mathbb{R}^d$. From this a change of variables shows that the inequalities

$$(10) \quad \int_{\mathbb{R}^k} \int_{\{|y_i| \sim 2^{n_i}\}} f(x) \chi_E(y; L_1(x, y), \dots, L_l(x, y)) dm_k(y) dm_k(x) \leq C \|f\|_{L^{d/k}(\mathbb{R}^k)} m_d(E)^{k/d}$$

hold uniformly in \mathbf{n} . This implies (8): suppose $E \subset [-2, 2]^d$. For a multi-index \mathbf{n} let $E_{\mathbf{n}}$ be the set of $(y_1, \dots, y_k; u_1, \dots, u_l) \in E$ for which $|y_i| \sim 2^{n_i}$. Then $m_d(E_{\mathbf{n}}) \leq 2^{\sum_i (n_i+1)}$ and so, if $\frac{1}{p} = \frac{k}{d} - \epsilon$,

$$m_d(E_{\mathbf{n}})^{k/d} \leq m_d(E)^{1/\tilde{p}} 2^{\epsilon \sum_i (n_i+1)}.$$

Applying (10) with E replaced by $E_{\mathbf{n}}$ and then summing over \mathbf{n} for which $-\infty < n_j \leq 0$ yields (8).

Moving to the proof of (9), we write, for suitable functions g on \mathbb{R}^d ,

$$(11) \quad \begin{aligned} \int_{\mathbb{R}^k} \int_{\{|y_i| \sim 1\}} f(x) g(y; L_1(x, y), \dots, L_l(x, y)) dm_k(y) dm_k(x) &= \langle T f, g \rangle \\ &= \int_{\mathbb{R}^l} \int_{\{|y_i| \sim 1\}} T f(y; u) g(y; u) dm_k(y) dm_l(u). \end{aligned}$$

Then (9) is a consequence of the fact, which we will establish below, that

$$(12) \quad T : L^{d/k}(\mathbb{R}^k) \rightarrow L^{d/l}(\mathbb{R}^d).$$

Although it does not figure here, one can regard the operator T as a restricted $(k-l)$ -plane transform operating on a function f defined on \mathbb{R}^k by integrating f over the $(k-l)$ -plane

$$\{x \in \mathbb{R}^k : L_1(x, y) = u_1, \dots, L_l(x, y) = u_l\}.$$

Since the indices in (12) are conjugate, it is natural to attempt to prove (12) by embedding T in an analytic family of operators $\{T_z\}$ and then interpolating between $L^1 \rightarrow L^\infty$ and $L^2 \rightarrow L^2$ estimates. Thus we define

$$T_z f(y; u) = C(z) f(y; \cdot) * |\cdot|^z(u),$$

where the convolution is in the u variable and $C(z)$ is chosen to compensate for the singularities of the distributions $|\cdot|^z$ on \mathbb{R}^l – see p. 363 in [3]. Next we will observe that

$$(13) \quad \|T_z f\|_{L^\infty(\mathbb{R}^d)} \leq c_0(y) \|f\|_{L^1(\mathbb{R}^k)}$$

if $z = 0 + is$ and then prove (using the hypothesis $(*)$) that

$$(14) \quad \|T_z f\|_{L^2(\mathbb{R}^d)} \leq c_1(y) \|f\|_{L^2(\mathbb{R}^k)}$$

if $z = -\frac{d}{2} + is$.

Note that (11) implies that $Tf(y; u) = 0$ unless $|y_j| \sim 1$. If $|y_j| \sim 1$ we will need the following formula:

$$(15) \quad \int_{\mathbb{R}^l} Tf(y; u)h(u) dm_l(u) = \int_{\mathbb{R}^k} f(x)h(L_1(x, y), \dots, L_l(x, y)) dm_k(x),$$

valid for nice functions h on \mathbb{R}^l . To see (15) with $y = \tilde{y}$, fix \tilde{y} with $|\tilde{y}_i| \sim 1$, take $g(y; u) = \chi_{B(\tilde{y}, \delta)}(y)h(u)$ in the extreme terms of (11), and then let $\delta \rightarrow 0$.

Now (13) follows immediately from (15).

To prove (14) we start by setting some notation. For fixed $y \in \mathbb{R}^k$ we consider the mapping \mathcal{L}_y of \mathbb{R}^k into \mathbb{R}^l defined by

$$\mathcal{L}_y x = (L_1(x, y), \dots, L_l(x, y))$$

along with the adjoint map \mathcal{L}_y^* of \mathbb{R}^l to \mathbb{R}^k defined by

$$\langle \mathcal{L}_y^* \zeta, x \rangle = \langle \zeta, \mathcal{L}_y x \rangle.$$

Then (15) implies that

$$(16) \quad \widehat{Tf(y, \cdot)}(\zeta) = \widehat{f}(\mathcal{L}_y^* \zeta).$$

In order to prove (14) by exploiting (16), we need a lemma.

Lemma. *Under the assumption (*) on \mathcal{C} , there is c , depending on \mathcal{C} and $\rho \in \mathbb{R}$, such that the inequality*

$$(17) \quad \int_{\{|y_j| \sim 1\}} \int_{\mathbb{R}^l} |\zeta|^\rho w(\mathcal{L}_y^* \zeta) dm_l(\zeta) dm_k(y) \leq c \int_{\mathbb{R}^k} |\tau|^{\rho-k+l} w(\tau) dm_k(\tau)$$

holds for nonnegative functions w on \mathbb{R}^k .

Proof of Lemma: If $x, y \in \mathbb{R}^k$, we may write $x(i)$ instead of x_i and xy to stand for the vector with $xy(i) = x(i)y(i)$. Let $\mathbf{1}$ stand for the vector $(1, 1, \dots, 1)$. One may check that, for $i = 1, \dots, k$, $\mathcal{L}_1^* \zeta(i) = \sum_j c_i^j \zeta_j$ and also that $\mathcal{L}_y^* \zeta = y \mathcal{L}_1^* \zeta$. In particular, the hypothesis (*) on \mathcal{C} has the following interpretation in terms of the $\binom{k}{l}$ coordinate projections π of \mathbb{R}^k onto \mathbb{R}^l : for each such π , $\pi \circ \mathcal{L}_1^* : \mathbb{R}^l \rightarrow \mathbb{R}^l$ is nonsingular. It follows that there is $M < \infty$ such that if $P \subset \{1, \dots, k\}$ satisfies $|P| = l$, then

$$(18) \quad |\zeta| \leq M \sup_{i \in P} |\mathcal{L}_1^* \zeta(i)|.$$

Next note that if $\zeta \in \mathbb{R}^l$, then there are $1 \leq i_{l+1} < i_{l+2} < \dots < i_k \leq k$, dependent on ζ , such that

$$(19) \quad |\zeta| \leq M |\mathcal{L}_1^* \zeta(i_a)| \text{ if } a = l+1, \dots, k.$$

(Having $|\zeta| > M |\mathcal{L}_1^* \zeta(i)|$ for even l i 's would contradict (18).) In this situation, write $Q = \{i_{l+1}, \dots, i_k\}$ and $\zeta \in F_Q$ so that $\mathbb{R}^l = \cup_Q F_Q$, where the union is taken over all $Q \subset \{1, \dots, k\}$ such that $|Q| = k-l$. Then (17) will follow by summing over Q the estimates

$$(20) \quad \int_{\{|y_i| \sim 1\}} \int_{F_Q} |\zeta|^\rho w(\mathcal{L}_y^* \zeta) dm_l(\zeta) dm_k(y) \leq c \int_{\mathbb{R}^k} |\tau|^{\rho-k+l} w(\tau) dm_k(\tau).$$

To establish (20), fix first $Q = \{i_{l+1}, \dots, i_k\}$, then i_1, \dots, i_l with $\{i_1, \dots, i_k\} = \{1, \dots, k\}$, and finally y_{i_1}, \dots, y_{i_l} with $|y_{i_a}| \sim 1$. Consider the map

$$(21) \quad (\zeta_1, \dots, \zeta_l, y_{i_{l+1}}, \dots, y_{i_k}) \mapsto \tau \doteq (y_{i_1} \mathcal{L}_1^* \zeta(i_1), \dots, y_{i_k} \mathcal{L}_1^* \zeta(i_k)) \simeq \mathcal{L}_y^* \zeta,$$

where the \simeq indicates a permutation of the coordinates. We want to estimate the absolute value J of the Jacobian determinant of (21) when $\zeta \in F_Q$. To do this, write τ as

$$\left(y_{i_1} \sum c_{i_1}^j \zeta_j, \dots, y_{i_l} \sum c_{i_l}^j \zeta_j, y_{i_{l+1}} \sum c_{i_{l+1}}^j \zeta_j, \dots, y_{i_k} \sum c_{i_k}^j \zeta_j \right),$$

where \sum means $\sum_{j=1}^l$. Computing the Jacobian matrix, one sees that

$$J = \prod_{a=1}^l |y_{i_a}| \times |D(i_1, \dots, i_l)| \times \prod_{a=l+1}^k |\mathcal{L}_1^* \zeta(i_a)|,$$

where $D(i_1, \dots, i_l)$ is the determinant of the l by l matrix obtained by retaining only the rows of \mathcal{C} corresponding to $i = i_1, \dots, i_l$. By (*), $|D(i_1, \dots, i_l)| \geq c(\mathcal{C}) > 0$. Since $\zeta \in F_Q$, $|\mathcal{L}_1^* \zeta(i_a)| \geq \frac{|\zeta|}{M}$ for $a = l+1, \dots, k$. It then follows from $|y_i| \sim 1$ that

$$(22) \quad J \geq c |\zeta|^{k-l}.$$

It is also easy to check (see (18)) that $|\zeta|^\rho \leq c |\mathcal{L}_1^* \zeta|^\rho$. So the inequality

$$\int_{\{|y_i| \sim 1\}} \int_{F_Q} |\zeta|^\rho w(\mathcal{L}_y^* \zeta) dm_l(\zeta) dy_{i_{l+1}} \cdots dy_{i_k} \leq c \int_{\mathbb{R}^k} |\tau|^{\rho-k+l} w(\tau) dm_k(\tau)$$

follows by change of variables, and then (20) follows by integrating with respect to y_{i_1}, \dots, y_{i_l} (since $|y_i| \sim 1$). This concludes the proof of the lemma.

With (17) we can prove (14): suppose $z = -\frac{d}{2} + is$. Then

$$\begin{aligned} \int_{\{|y_i| \sim 1\}} \int_{\mathbb{R}^l} |T_z f(y, u)|^2 dm_l(u) dm_k(y) &= \int_{\{|y_i| \sim 1\}} \int_{\mathbb{R}^l} |\widehat{T_z f(y, \cdot)}(\zeta)|^2 dm_l(\zeta) dm_k(y) \\ &= c(s) \int_{\{|y_i| \sim 1\}} \int_{\mathbb{R}^l} |\widehat{f}(\mathcal{L}_y^* \zeta)|^2 |\zeta|^{d/2-l} dm_l(\zeta) dm_k(y) \\ &\leq c(s) \int_{\mathbb{R}^k} |\widehat{f}(\tau)|^2 |\tau|^{d-2l-(k-l)} dm_k(\tau) = c(s) \|f\|_{L^2(\mathbb{R}^k)}^2, \end{aligned}$$

where: the second equality follows from (16) and the fact that, on \mathbb{R}^l , $|\widehat{\cdot}^z(\zeta)| = c(z) |\zeta|^{-z-l}$ ([3], p. 363); the inequality follows from (17); and the last equality follows from $d = k + l$. This proves (14). Now interpolating between (13) and (14) shows that

$$T_z : L^{d/k}(\mathbb{R}^k) \rightarrow L^{d/l}(\mathbb{R}^d)$$

if $z = -l + is$. Since T_{-l} is a scalar multiple of T , (12) follows, concluding the proof of the theorem.

REFERENCES

- [1] M. Christ, Convolution, curvature, and combinatorics: a case study, *Internat. Math. Res. Notices* **19** (1998), 1033–1048.
- [2] S.W. Drury and K. Guo, Convolution estimates related to surfaces of half the ambient dimension, *Math. Proc. Cambridge Phil. Soc.* **110** (1991), 151–159.
- [3] I.M. Gelfand and G.E. Shilov, Generalized Functions, vol. I, Academic Press, New York, 1964.
- [4] D. Oberlin, Convolution and restriction for a 3-surface in \mathbb{R}^5 , *J. of Fourier Analysis and Applications* **10** (2004), pp. 377–382.
- [5] F. Ricci, L^p - L^q boundedness for convolution operators defined by singular measures in \mathbb{R}^n , *Boll. Un. Mat. Ital. A* **11** (1997), 237–252.
- [6] E.M. Stein, Harmonic analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, New Jersey, 1993.

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